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CR-INVARIANTS AND THE SCATTERING OPERATOR FOR COMPLEX MANIFOLDS WITH BOUNDARY

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Contents

1. Introduction

The purpose of this paper is to describe certain CR-covariant differential operators on a strictly pseudoconvex CR manifold M as residues of the scattering operator for the Laplacian on an ambient complex Kähler manifold X having M as a 'CR-infinity.' We also characterize the CR Q-curvature in terms of the scattering operator. Our results parallel earlier results of Graham and Zworski [14], who showed that if X is an asymptotically hyperbolic manifold carrying a Poincaré-Einstein metric, the Q-curvature and certain conformally covariant differential operators on the 'conformal infinity' M of X can be recovered from the scattering operator on X. The results in this paper were announced in [18].

To describe our results, we first recall some basic notions of CR geometry and recent results [8], [9] concerning CR-covariant differential operators and CR-analogues of Q-curvature. If M is a smooth, orientable manifold of real dimension $(2n+1)$, a *CR-structure* on M is a real hyperplane bundle H on TM together with a smooth

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bundle map $J : H \to H$ with $J^2 = -1$ that determines an almost complex structure on H. We denote by $T_{1,0}$ the eigenspace of J on $H \otimes \mathbb{C}$ with eigenvalue $+i$; we will always assume that the CR-structure on M is integrable in the sense that $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. We will assume that M is orientable, so that the line bundle $H^\perp \subset T^*M$ admits a nonvanishing global section. A *pseudo-Hermitian structure* on M is smooth, nonvanishing section θ of $H^\perp$. The Levi form of θ is the Hermitian form $L_\theta(v, w) = d\theta(v, Jw)$ on H. The CR structure on M is called *strictly pseudoconvex* if the Levi form is positive definite. Note that this condition is actually independent of the choice of θ compatible with a given orientation of M. We will always assume that M is strictly pseudoconvex in what follows. It follows from strict pseudoconvexity that θ is a contact form, and the form $\theta \wedge (d\theta)^n$ is a volume form that defines a natural inner product on $\mathcal{C}^\infty(M)$ by integration. The pseudo-Hermitian structure on M also determines a connection on TM, the Tanaka-Webster connection ∇_θ; the basic data of pseudo-Hermitian geometry are the curvature and torsion of this connection (see [27], [29]).

Given a fixed CR-structure (H, J) on M, any nonvanishing section $\overline{\theta}$ of $H^\perp$ compatible with a given orientation takes the form $e^{2\Upsilon}\theta$ for a fixed section θ of $H^\perp$ and some function $\Upsilon \in \mathcal{C}^\infty(M)$. The corresponding Levi form is given by $L_{\overline{\theta}} = e^{2\Upsilon} L_\theta$. In this sense the CR-structure determines a conformal class of pseudo-Hermitian structures on M.

For strictly pseudoconvex CR-manifolds, Fefferman and Hirachi [8] proved the existence of CR-covariant differential operators P_k of order $2k$, $k = 1, 2, \ldots, n+1$, whose principal parts are Δ_θ^k, where Δ_θ is the sub-Laplacian on M with respect to the pseudo-Hermitian structure θ. They exploit Fefferman's construction (formulated intrinsically by Lee in [21]) of a circle bundle $\mathcal{C}$ over M with a natural conformal structure and a mapping $\theta \mapsto g_\theta$ from pseudo-Hermitian structures on M to Lorentz metrics on $\mathcal{C}$ that respects conformal classes. They then construct the conformally covariant differential operators found in [12] (referred to here as GJMS operators) on $\mathcal{C}$, and show that these operators pull back to CR-covariant differential operators on M. The CR Q-curvature may be similarly defined as a pullback to M of Branson's Q-curvature on the circle bundle $\mathcal{C}$. Here we will show that the operators P_k on M occur as residues for the scattering operator associated to a natural scattering problem with M as the boundary at infinity, and that the CR Q-curvature Q_θ^{CR} can be computed from the scattering operator.

To describe the scattering problem, we first discuss its geometric setting. Recall that if M is an integrable, strictly pseudoconvex CR-manifold of dimension $(2n+1)$ with $n \geq 2$, there is a complex manifold X of complex dimension $m = n+1$ having M as its boundary so that the CR-structure on M is induced from the complex structure on X (this result is false, in general, when $n = 1$; see [17]). Let φ be a defining function for M and denote by $\mathring{X}$ the interior of X (we take $\varphi < 0$ in $\mathring{X}$). The associated Kähler metric g on $\mathring{X}$ is the Kähler metric with Kähler form

$$\omega_\varphi = -\frac{i}{2}\partial\overline{\partial}\log(-\varphi) \tag{1.1}$$

in a neighborhood of M, extended smoothly to all of X. The metric has the form

$$g_\varphi = -\frac{\eta}{\varphi} + (1 - r\varphi)\left(\frac{d\varphi^2}{\varphi^2} + \frac{\Theta^2}{\varphi^2}\right). \tag{1.2}$$

in a neighborhood of M, where η and Θ have Taylor series to all orders in φ at $\varphi = 0$. The boundary values $\Theta|_M = \theta$, and $\eta|_H = h$ induce respectively a contact form on M and a Hermitian metric on H. The function r is a smooth function, the transverse curvature, which depends on the choice of φ (see [13]). Thus, the conformal class of a Hermitian metric h on H, a *subbundle* of TM, is a kind of 'Dirichlet datum at infinity' for the metric g_φ, that is $(-\varphi)g_\varphi|_H = h$.

A motivating example for our work is the case of a strictly pseudoconvex domain $X \subset \mathbb{C}^m$ with Hermitian metric

$$g = \sum_{j,k=1}^{m} \frac{\partial^2}{\partial z_j \partial z_{\overline{k}}} \log\left(-\frac{1}{\varphi}\right) dz_j \otimes dz_{\overline{k}},$$

where φ is a defining function for the boundary of X with $\varphi < 0$ in the interior of X. In this example, observe that if

$$\Theta = \frac{i}{2}\left(\overline{\partial}\varphi - \partial\varphi\right)$$

and $\iota : M \to X$ is the natural inclusion, then $\theta = \iota^*\Theta$ is a contact form on M that defines the CR-structure $H = \ker\theta$. The form $d\theta$ induces the Levi form on M and so defines a pseudo-Hermitian structure on M. Denote by J the almost complex structure on H; the two-form $h = d\theta(\,\cdot\,, J\,\cdot\,)$ is a pseudo-Hermitian metric on M. It is not difficult to see that the conformal class of the pseudo-Hermitian structure on M, i.e., its CR-structure, is independent of the choice of defining function φ.

It is natural to consider scattering theory for the Laplacian, Δ_g, on $(\mathring{X}, g)$, where X is a complex manifold with boundary M. As discussed in what follows, the metric g belongs to the class of Θ-metrics considered by Epstein, Melrose, and Mendoza [4]; see also the recent paper of Guillarmou and Sá Barreto [15] where scattering theory for asymptotically complex hyperbolic manifolds (a class which includes those considered here) is analyzed in depth. Thus, the full power of the Epstein-Melrose-Mendoza analysis of the resolvent $R(s) = (\Delta_g - s(m-s))^{-1}$ of Δ_g is available to study scattering theory on $(\mathring{X}, g)$.

For $f \in \mathcal{C}^\infty(M)$, $\Re(s) = m/2$, and $s \neq m/2$, there is a unique solution u of the 'Dirichlet problem'

$$\begin{aligned} (\Delta_g - s(m-s))\, u &= 0 \\ u &= (-\varphi)^{m-s} F + (-\varphi)^s G \\ F|_M &= f. \end{aligned} \tag{1.3}$$

where $F, G \in \mathcal{C}^\infty(X)$. The uniqueness follows from the absence of L^2 solutions of the eigenvalue problem for $\Re(s) = m/2$; this may be proved, for example, using [28] (see the comments in [15]). Here we will use the explicit formulas for the Kähler form and Laplacian obtained in [13] to obtain the asymptotic expansions of solutions to the generalized eigenvalue problem.

Unicity for the 'Dirichlet problem' (1.3) implies that the Poisson map

$$\begin{aligned} \mathcal{P}(s) : \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(\mathring{X}) \\ f &\mapsto u \end{aligned} \tag{1.4}$$

and the scattering operator

$$\begin{aligned} S_X(s) : \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M) \\ f &\mapsto G|_M \end{aligned}$$

are well-defined. The operator $S_X(s)$ depends *a priori* on the boundary defining function φ for M. If $\overline{\varphi} = e^{\upsilon}\varphi$ is another defining function for M and $\upsilon|_M = \Upsilon$, the corresponding scattering operator $\overline{S}_X(s)$ is given by

$$\overline{S}_X(s) = e^{-s\Upsilon} S_X(s) e^{(s-m)\Upsilon}.$$

The operator $S_X(s)$ admits a meromorphic continuation to the complex plane, possibly with singularities at $s = 0, -1, -2, \cdots$; see [25] where the scattering operator is described and the problem of studying its poles and residues is posed, and see [15] for a detailed analysis of the scattering operator. The scattering operator is self-adjoint for s real. We will show that, with a geometrically natural choice of the boundary defining function φ, the residues of certain poles of $S_X(s)$ are CR-covariant differential operators.

To describe the setting for this result, recall that for strictly pseudoconvex domains Ω in $\mathbb{C}^m$, Fefferman [6] proved the existence of a defining function φ for $\partial\Omega$ which is an approximate solution of the complex Monge-Ampère equation.

The complex Monge-Ampère equation for a function $\varphi \in \mathcal{C}^\infty(\Omega)$ is the equation

$$\begin{aligned} J[\varphi] &= 1 \\ \varphi|_{\partial\Omega} &= 0 \end{aligned}$$

where J is the complex Monge-Ampère operator

$$J[\varphi] = \det \begin{bmatrix} \varphi & \varphi_j \\ \varphi_{\overline{k}} & \varphi_{j\overline{k}} \end{bmatrix}$$

We say that $\varphi \in \mathcal{C}^\infty(\Omega)$ is an approximate solution of the complex Monge-Ampère equation if

$$J[\varphi] = 1 + \mathcal{O}\left(\varphi^{m+1}\right)$$

$$\varphi|_{\partial\Omega} = 0$$

The Kähler metric g associated to such an approximate solution φ is an approximate Kähler-Einstein metric on Ω, i.e., g obeys

$$\operatorname{Ric}(g) = -(m+1)\omega + \mathcal{O}(\varphi^{m-1}). \tag{1.5}$$

where ω is the Kähler form associated to φ, and Ric is the Ricci form.

Under certain conditions, Fefferman's result can be 'globalized' to the setting of complex manifolds X with strictly pseudoconvex boundary M, as we discuss below. It follows that $\mathring{X}$ carries an approximate Kähler-Einstein metric g in the sense that (1.5) holds.

We will call a smooth function φ defined in a neighborhood of M a *globally defined approximate solution* of the Monge-Ampère equation on X if for each $p \in M$ there is a neighborhood U of p in X and a holomorphic coordinate system in U for which φ is an approximate solution of the Monge-Ampère equation. As we will show, such a solution exists if and only if M admits a pseudo-Hermitian structure θ which is volume-normalized with respect to some locally defined, closed $(n+1,0)$-form in a neighborhood of any point $p \in M$ (see section 2.4.2 where we defined

"volume-normalized", and see Burns-Epstein [2] where a similar condition is used to construct a global solution of the Monge-Ampére equation when $\dim M = 3$). If $\dim M \geq 5$, we can give a more geometric formulation of this condition. Recall that a CR-manifold is *pseudo-Einstein* if there is a pseudo-Hermitian structure θ for which the Webster Ricci curvature is a multiple of the Levi form (see Lee [22] where this geometric notion is introduced and studied). In [22], Lee proved that if $\dim M \geq 5$, then M admits a pseudo-Einstein, pseudo-Hermitian structure θ if and only if θ is volume-normalized with respect to a closed $(n+1,0)$-form in a neighborhood of any point $p \in M$. If $\dim M = 3$, the pseudo-Einstein condition is vacuous and must be replaced by a more stringent condition; see section 2.4.2 in what follows. If X is a pseudoconvex domain in $\mathbb{C}^m$, this condition is trivially satisfied since the pseudo-Hermitian structure induced by the Fefferman approximate solution is volume-normalized with respect to the restriction of $\zeta = dz^1 \wedge \cdots \wedge dz^m$ to M.

Theorem 1.1. *Let X be a complex manifold of complex dimension $m = n+1$ with strictly pseudoconvex boundary M. Let g be the Kähler metric on X associated to the Kähler form (1.1), and let $S_X(s)$ be the scattering operator for Δ_φ. Finally, suppose that Δ_φ has no L^2-eigenvalues. Then $S_X(s)$ has simple poles at the points $s = m/2 + k/2$, $k \in \mathbb{N}$, and*

$$\operatorname*{Res}_{s=m/2+k/2} S_X(s) = c_k P_k, \tag{1.6}$$

where the P_k are differential operators of order $2k$, and

$$c_k = \frac{(-1)^k}{2^k k!(k-1)!}. \tag{1.7}$$

If g is an approximate Kähler-Einstein metric given by a globally defined approximate solution of the Monge-Ampère equation, then for $1 \leq k \leq m$, the operators P_k are CR-covariant differential operators.

Remark 1.2. *It is not difficult to show that, for generic compactly supported perturbations of the metric, L^2-eigenvalues are absent. Our analysis applies if only the metric g has the form (1.2) in a neighborhood of M.*

Remark 1.3. *We view the operators P_k as operators on $\mathcal{C}^\infty(M)$; if one instead views these operators as acting on appropriate density bundles over M they are actually invariant operators. Gover and Graham* [9] *showed that the CR-covariant differential operators P_k are logarithmic obstructions to the solution of the Dirichlet problem (1.3) when X is a pseudoconvex domain in $\mathbb{C}^m$ with a metric of Bergman type, but did not identify them as residues of the scattering operator.*

It follows from the self-adjointness (s real) and conformal covariance of $S_X(s)$ that the operators P_k are self-adjoint and conformally covariant. As in [14], the analysis centers on the Poisson map $\mathcal{P}(s)$ defined in (1.4). As shown in [4], the Poisson map is analytic in s for $\mathrm{Re}(s) > m/2$. Moreover, at the points $s = m/2 + k/2$, $k = 1, 2, \cdots$, the Poisson operator takes the form

$$\mathcal{P}(s)f = (-\varphi)^{m/2-k/2} F + [(-\varphi)^{m/2+k/2} \log(-\varphi)] G$$

for functions $F, G \in \mathcal{C}^\infty(X)$ with

$$F|_M = f, \ G|_M = c_k P_k f.$$

Here P_k are differential operators determined by a formal power series expansion of the Laplacian (see Lemma 3.4), and are the same operators that appear as residues of the scattering operator at points $s = m/2 + k/2$. An important ingredient in the analysis is the asymptotic form of the Laplacian due to Lee and Melrose [23] and refined by Graham and Lee in [13].

If the defining function φ is an approximate solution of the complex Monge-Ampère equation, the differential operators P_k, $1 \leq k \leq m$, can be identified with the GJMS operators owing to the characterization of $\mathcal{P}(s)f$ described above (see Proposition 5.4 in [9]; the argument given there for pseudoconvex domains easily generalizes to the present setting).

Explicit computation shows that, for an approximate Kähler-Einstein metric g, the first operator has the form

$$P_1 = c_1(\Delta_b + n(2(n+1))^{-1}R),$$

where Δ_b is the sub-Laplacian on X and R is the Webster scalar curvature, i.e., P_1 is the CR-Yamabe operator of Jerison and Lee [19].

The CR Q-curvature is a pseudo-Hermitian invariant realized as the pullback to M of the Q-curvature of the circle bundle $\mathcal{C}$.

Theorem 1.4. *Suppose that X is a complex manifold with strictly pseudoconvex boundary M, and suppose that g is an approximate Kähler-Einstein metric given by a globally defined approximate solution of the Monge-Ampère equation. Let $S_X(s)$ be the associated scattering operator. The formula*

$$c_m Q_\theta^{CR} = \lim_{s \to m} S_X(s)1$$

holds, where c_m is given by (1.7).

It follows from Theorem 1.1 and the conformal covariance of $S_X(s)$ that if $\overline{\theta} = e^{2\Upsilon}\theta$, then

$$e^{2m\Upsilon} Q_{\overline{\theta}}^{CR} = Q_\theta^{CR} + P_m \Upsilon$$

as was already shown in Fefferman-Hirachi [8]. From this it follows that the integral $\int_M Q_\theta^{CR}\ \psi$ is a CR-invariant (recall that ψ is the natural volume form on M defined by the contact form θ). We remark that the integral of Q_θ^{CR} vanishes for any three-dimensional CR-manifold because the integrand is a total divergence (see [8], Proposition 3.2 and comments below), while under the condition of our Theorem 1.4, there is a pseudo-Hermitian structure for which $Q_\theta^{CR} = 0$ (see [8], Proposition 3.1). In our case, if φ is a globally defined approximate solution of the Monge-Ampère equation, the induced contact form $\theta = (i/2)\,(\overline{\partial}\varphi - \partial\varphi)$ on M is an 'invariant contact form' in the language of [8], and they show in Proposition 3.1 that $Q_\theta^{CR} = 0$ for an invariant contact form. Thus it is not clear at present under what circumstances this invariant is nontrivial for a general, strictly pseudoconvex manifold.

Finally, we prove a CR-analogue of Graham and Zworski's result ([14], Theorem 3) using scattering theory.

Theorem 1.5. *Suppose that X is a compact complex manifold with strictly pseudoconvex boundary M, and g is an approximate-Kähler-Einstein metric given by a globally defined approximate solution of the Monge-Ampère equation. Then*

$$\text{(1.8)} \quad \operatorname{vol}_g \{-\varphi > \varepsilon\} = c_0 \varepsilon^{-n-1} + c_1 \varepsilon^{-n} + \cdots + c_n \varepsilon^{-1} + L \log(-\varepsilon) + V + o(1).$$

where

$$L = c_m \int_M Q_\theta^{CR}\ \psi = 0$$

We remark that Seshadri [26] already showed that L is, up to a constant, the integral of Q_θ^{CR}. It is worth noting that our choice of defining function differs from Seshadri's.

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2. Geometric Preliminaries

2.1. **CR Manifolds.** Suppose that M is a smooth orientable manifold of real dimension $2n+1$, and let $\mathbb{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle on M. A *CR-structure* on M is a complex n-dimensional subbundle $\mathcal{H}$ of $\mathbb{C}TM$ with the property that $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$. If, also, $[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}$, we say that the CR-structure is *integrable*. If we set $H = \operatorname{Re} \mathcal{H}$, then the bundle H has real codimension one in TM. The map

$$\begin{gathered} J : H \to H \\ V + \overline{V} \mapsto i(V - \overline{V}) \end{gathered}$$

satisfies $J^2 = -I$ and gives H a natural complex structure.

Since M is orientable, there is a nonvanishing one-form θ on M with $\ker \theta = H$. This form is unique up to multiplication by a positive, nonvanishing function $f \in \mathcal{C}^\infty(M)$. A choice of such a one-form θ is called a *pseudo-Hermitian structure* on M. The *Levi form* is given by

$$L_\theta(V, \overline{W}) = -i d\theta(V, \overline{W}). \tag{2.1}$$

for $V, W \in \mathcal{H}$ (here $d\theta$ is extended to $\mathcal{H}$ by complex linearity). Note that

$$L_{f\theta} = f L_\theta \tag{2.2}$$

since θ annihilates $\mathcal{H}$. If $d\theta$ is nondegenerate, then there is a unique real vector field T on M, the *characteristic vector field* T, with the properties that $\theta(T) = 1$ and $T \lrcorner\, d\theta = 0$. If $\{W_\alpha\}$ is a local frame for $\mathcal{H}$ (here α ranges from 1 to n), then the vector fields $\{W_\alpha, W_{\overline{\alpha}}, T\}$ form a local frame for $\mathbb{C}TM$. If we choose $(1,0)$-forms θ^α dual to the W_α then $\left\{\theta^\alpha, \theta^{\overline{\alpha}}, \theta\right\}$ forms a dual coframe for $\mathbb{C}TM$. We say that $\{\theta^\alpha\}$ forms an admissible coframe dual to $\{W^\alpha\}$ if $\theta^\alpha(T) = 0$ for all α. The integrability condition is equivalent to the condition that

$$d\theta = d\theta^\alpha = 0 \mod \{\theta, \theta^\alpha\} \tag{2.3}$$

The Levi form is then given by

$$L_\theta = h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} \tag{2.4}$$

for a Hermitian matrix-valued function $h_{\alpha\overline{\beta}}$. We will use $h_{\alpha\overline{\beta}}$ to raise and lower indices in this article.

We will say that a given CR-structure is *strictly pseudoconvex* if L_θ is positive definite. Note that (up to sign) this condition is independent of the choice of pseudo-Hermitian structure θ.

In what follows, we will always suppose that M is orientable and that M carries a strictly pseudoconvex, integrable CR-structure. In this case, the pseudo-Hermitian geometry of M can be understood in terms of the *Tanaka-Webster connection* on M (see Tanaka [27] and Webster [29]). With respect to the frame discussed above, the Tanaka-Webster connection is given by

$$\nabla W_\alpha = \omega_\alpha^{\ \beta} \otimes W_\beta, \ \nabla T = 0 \tag{2.5}$$

for connection one-forms $\omega_\alpha^{\ \beta}$ obeying the structure equations

$$d\theta^\alpha = \theta^\beta \wedge \omega_\alpha^{\ \beta} + \theta \wedge \tau^\alpha$$
$$d\theta = i h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}}$$

where the torsion one-forms are given by

$$\tau^\alpha = A^\alpha_{\overline{\beta}} \theta^{\overline{\beta}},$$

with $A_{\alpha\beta} = A_{\beta\alpha}$. The connection obeys the compatibility condition

$$dh_{\alpha\overline{\beta}} = \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha}.$$

with the Levi form described in (2.1) and (2.4).

2.2. **Complex Manifolds with CR Boundary.** Now suppose that X is a compact complex manifold of dimension $m = n+1$ with boundary $\partial X = M$. We will denote by $\mathring{X}$ the interior of X. The manifold M inherits a natural CR-structure from the complex structure of the ambient manifold. We will suppose that that M is strictly pseudoconvex; such a structure, induced by the complex structure of the ambient manifold, is always integrable.

We will suppose that $\varphi \in \mathcal{C}^\infty(X)$ is a defining function for M, i.e., $\varphi < 0$ in $\mathring{X}$, $\varphi = 0$ on M, and $d\varphi(p) \neq 0$ for all $p \in M$. We will further suppose that φ has no critical points in a collar neighborhood of M so that the level sets $M^\varepsilon = \varphi^{-1}(-\varepsilon)$ are smooth manifolds for all ε sufficiently small.

Associated to the defining function φ is the Kähler form

$$\omega_\varphi = -\frac{i}{2}\partial\overline{\partial}\log(-\varphi) = \frac{i}{2}\left(\frac{\partial\overline{\partial}\varphi}{-\varphi} + \frac{\partial\varphi \wedge \overline{\partial}\varphi}{\varphi^2}\right) \tag{2.6}$$

We will study scattering on X with the metric induced by the Kähler form (2.6). Since we can cover a neighborhood of M in X by coordinate charts, it suffices to consider the situation where U is an open subset of $\mathbb{C}^m$ and $\varphi : U \to \mathbb{R}$ is a smooth function with no critical points in U, the set $\{\varphi < 0\}$ is biholomorphically equivalent to a boundary neighborhood in X, and $\{\varphi = 0\}$ is diffeomorphic to the corresponding boundary neighborhood in M. We will now describe the asymptotic geometry near M, recalling the ambient metric of [13] and computing the asymptotic form of the metric and volume form.

The manifolds M^ε inherit a natural CR-structure from the ambient manifold X with

$$\mathcal{H}^\varepsilon = \mathbb{C}TM^\varepsilon \cap T^{1,0}U.$$

Given a defining function φ, we define a one-form

$$\Theta = \frac{i}{2}\left(\overline{\partial} - \partial\right)\varphi$$

and let

$$\theta_\varepsilon = \iota_\varepsilon^* \Theta$$

where $\iota_\varepsilon : M^\varepsilon \to U$ is the natural embedding. The contact form θ_ε gives M^ε a pseudo-Hermitian structure. We will denote by $\mathcal{H}$ the subbundle of $T^{1,0}U$ whose fibre over M^ε is $\mathcal{H}^\epsilon$. Note that

$$d\Theta = i\partial\overline{\partial}\varphi.$$

and the Levi form on M^ε is given by

$$L_{\theta_\varepsilon} = -id\theta_\varepsilon$$

We will assume that each M^ε is strictly pseudoconvex, i.e., L_{θ_ε} is positive definite for all sufficiently small $\varepsilon > 0$. To simplify notation, we will write θ for θ_ϵ, suppressing the ϵ, as the meaning will be clear from the context.

2.2.1. *Ambient Connection.* In order to describe the asymptotic geometry of X, we recall the ambient connection defined by Graham and Lee [13] that extends the Tanaka-Webster connection on each M^ε to $\mathbb{C}TU$. First we recall the following lemma from [23].

Lemma 2.1. *There exists a unique* $(1,0)$*-vector field* ξ *on* U *so that:*

$$\partial\varphi(\xi) = 1 \tag{2.7}$$

and

$$\xi \lrcorner \partial\overline{\partial}\varphi = r\ \overline{\partial}\varphi \tag{2.8}$$

for some $r \in \mathcal{C}^\infty(U)$.

The smooth function r in (2.8) is called the *transverse curvature.*

We decompose ξ into real and imaginary parts,

$$\xi = \frac{1}{2}(N - iT), \tag{2.9}$$

where N and T are real vector fields on U. It easily follows from (2.9) that

$$d\varphi(N) = 2,\ \theta(N) = 0$$

and

$$\theta(T) = 1,\ T \lrcorner d\theta = 0.$$

Thus T is the characteristic vector field for each M^ε, and N is normal to each M^ε.

Let $\{W_\alpha\}$ be a frame for $\mathcal{H}$. It follows from Lemma 2.1 that $\{W_\alpha, W_{\overline{\alpha}}, T\}$ forms a local frame for $\mathbb{C}TM^\varepsilon$, while $\left\{W_\alpha, W_{\overline{\alpha}}, \xi, \overline{\xi}\right\}$ forms a local frame for $\mathbb{C}TU$. If $\{\theta^\alpha\}$ is a dual coframe for $\{W_\alpha\}$, then $\left\{\theta^\alpha, \theta^{\overline{\alpha}}, \theta\right\}$ is a dual coframe for $\mathbb{C}TM^\varepsilon$, while $\left\{\theta^\alpha, \theta^{\overline{\alpha}}, \partial\varphi, \overline{\partial}\varphi\right\}$ is a dual coframe for $\mathbb{C}TU$. The Levi form on each $\mathcal{H}^\varepsilon$ is given by

$$L_\theta = h_{\alpha\overline{\beta}}\,\theta^\alpha \wedge \theta^{\overline{\beta}}$$

for a Hermitian matrix-valued function $h_{\alpha\overline{\beta}}$. We will use $h_{\alpha\overline{\beta}}$ to raise and lower indices. We will set

$$W_m = \xi,\ W_{\overline{m}} = \overline{\xi},\ ,\theta^m = \partial\varphi,\ \theta^{\overline{m}} = \overline{\partial}\varphi.$$

In what follows, repeated Greek indices are summed from 1 to n and repeated Latin indices are summed from 1 to $m = n+1$.

The following important lemma decomposes the form $d\Theta$ into 'tangential' and 'transverse' components.

Lemma 2.2. *The formula*

$$\partial\overline{\partial}\varphi = h_{\alpha\overline{\beta}}\ \theta^\alpha \wedge \theta^{\overline{\beta}} + r\ \partial\varphi \wedge \overline{\partial}\varphi$$

holds.

Graham and Lee [13] proved:

Proposition 2.3. *There exists a unique linear connection ∇ on U so that*

- **(a):** *For any vector fields X and Y on U tangent to some M^ε, $\nabla_X Y = \nabla^\varepsilon_X Y$ where ∇^ε is the pseudo-Hermitian connection on M^ε.*
- **(b):** *∇ preserves $\mathcal{H}$, N, T, and L_θ; that is, $\nabla_X \mathcal{H} \subset \mathcal{H}$ for any $X \in \mathbb{C}TU$, and $\nabla T = \nabla N = \nabla L_\theta = 0$.*
- **(c):** *If $\{W_\alpha\}$ is a frame for $\mathcal{H}$, and $\{\theta^\alpha, \partial\varphi\}$ is the dual $(1,0)$-coframe on U, then*

$$d\theta^\alpha = \theta^\beta \wedge \varphi_\beta{}^\alpha - i\partial\varphi \wedge \tau^\alpha + i(W^\alpha r) d\varphi \wedge \theta + \frac{1}{2} r\ d\varphi \wedge \theta^\alpha \tag{2.10}$$

The connection ∇ is called the *ambient connection.*

2.2.2. *Kähler Metric.* Using Lemma 2.2, we can also compute the Kähler form

$$\omega_\varphi = \frac{i}{2}\left(\frac{1}{-\varphi} h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} + \frac{1 - r\varphi}{\varphi^2} \partial\varphi \wedge \overline{\partial}\varphi \right). \tag{2.11}$$

The induced Hermitian metric is

$$g_\varphi = \frac{1}{-\varphi} h_{\alpha\overline{\beta}} \theta^\alpha \otimes \theta^{\overline{\beta}} + \frac{1 - r\varphi}{\varphi^2} \partial\varphi \otimes \overline{\partial}\varphi. \tag{2.12}$$

It is easily computed that

$$g_\varphi(N, N) = 4 \frac{1 - r\varphi}{\varphi^2}$$

so that the outward unit normal field associated to the surfaces M^ε is

$$\nu = \frac{-\varphi}{2\sqrt{1 - r\varphi}} N \tag{2.13}$$

We note for later use that the induced volume form ω_φ^m is given by

$$\omega_\varphi^m = \left(\frac{i}{2}\right)^m \left(\frac{1 - r\varphi}{(-\varphi)^{m+1}} \det\left(h_{\alpha\overline{\beta}}\right)\ \theta^1 \wedge \theta^{\overline{1}} \wedge \cdots \wedge \theta^m \wedge \theta^{\overline{m}} \right) \tag{2.14}$$

while

$$\nu \lrcorner \left.\omega_\varphi^m\right|_{M^\varepsilon} = \frac{m}{2^{n-1}} \frac{1 - r\varepsilon}{\varepsilon^m} \left(d\theta_\varepsilon\right)^n \wedge \theta_\varepsilon \tag{2.15}$$

We will set

$$\psi = \frac{m}{2^{n-1}} \left(d\theta\right)^n \wedge \theta \tag{2.16}$$

We also note for later use that if $u \in \mathcal{C}^\infty(X)$ and

$$du = u_\alpha \theta^\alpha + u_{\overline{\alpha}} \theta^{\overline{\alpha}} + u_m \partial\varphi + u_{\overline{m}} \overline{\partial}\varphi \tag{2.17}$$

then

$$|du|^2_{g_\varphi} = -\varphi h^{\alpha\overline{\beta}} u_\alpha u_{\overline{\beta}} + \frac{\varphi^2}{1 - r\varphi} u_m u_{\overline{m}} \tag{2.18}$$

2.3. **The Laplacian on X.** The Laplacian on the Kähler manifold (X, ω_φ) is the operator[1]

$$\begin{aligned}\Delta_\varphi u &= \operatorname{Tr}\left(i\partial\overline{\partial}u\right) \\ &= g^{j\overline{k}} u_{j\overline{k}}\end{aligned} \tag{2.19}$$

for $u \in \mathcal{C}^\infty(X)$, where we now write Δ_φ rather than Δ_g to emphasize the dependence of Δ on the boundary defining function φ.

Graham and Lee [13] computed the Laplacian in a collar neighborhood of M, separating 'normal' and 'tangential' parts. To state their results, recall that the sub-Laplacian is defined on each M^ε by

$$\Delta_b u = \left(u_\alpha{}^\alpha + u_{\overline{\beta}}{}^{\overline{\beta}}\right) \tag{2.20}$$

where covariant derivatives are taken with respect to the Tanaka-Webster connection on M^ε.

Graham and Lee [13] proved:

Theorem 2.4. *The formula*

$$\Delta_\varphi = \frac{\varphi}{4}\left[\frac{-\varphi}{1 - r\varphi}\left(N^2 + T^2 + 2rN + 2X_r\right) - 2\Delta_b + 2nN\right] \tag{2.21}$$

holds, where

$$X_r = r^\alpha W_\alpha + r^{\overline{\alpha}} W_{\overline{\alpha}}.$$

It will be useful to recast the above formula for $\Delta_\varphi u$ in terms of $x = -\varphi$. Note that

$$N = 2\frac{\partial}{\partial\varphi} = -2\frac{\partial}{\partial x} \tag{2.22}$$

so that

$$\begin{aligned}\Delta_\varphi u = &\left(\frac{1}{1 + rx}\right)\left(x\frac{\partial}{\partial x}\right)^2 u - (n+1)x\frac{\partial}{\partial x} \\ &+ \frac{1}{4}\left(\frac{x^2}{1 + rx}\right)\left(T^2 u - 2ru_x + 2X_r u\right) \\ &+ \frac{1}{4}x\left(-2\Delta_b u\right)\end{aligned}$$

We think of Δ_φ as a variable-coefficient differential operator with respect to vector fields $(x\partial_x)$ and vector fields tangent to the boundary M. In a neighborhood of M we have

$$\Delta_\varphi \sim \sum_{k \geq 0} x^k L_k \tag{2.23}$$

for differential operators L_k, where the indicial operator L_0 is

$$L_0 = -\left(\left(x\frac{\partial}{\partial x}\right)^2 - mx\frac{\partial}{\partial x}\right) \tag{2.24}$$

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[1] Note that our definition differs from that of Graham and Lee by an overall factor of -1/4.

and the operator L_1 is

$$L_1 = \frac{1}{4}\left(-2\Delta_b u - 4r_0 x\frac{\partial}{\partial x} - 4r_0\left[\left(x\frac{\partial}{\partial x}\right)^2 - x\frac{\partial}{\partial x}\right]\right) \tag{2.25}$$

where

$$r = r_0 + \mathcal{O}(x).$$

2.4. **The Complex Monge-Ampère Equation.**

2.4.1. *Local Theory.* Let Ω be a domain in $\mathbb{C}^m$ with smooth boundary. The complex Monge-Ampère equation is the nonlinear equation

$$J[u] = 1$$
$$u|_{\partial\Omega} = 0$$

for a function $u \in \mathcal{C}^\infty(\Omega)$, $u > 0$ on Ω, where $J[u]$ is the Monge-Ampère operator:

$$J[u] = (-1)^m \det\begin{pmatrix} u & u_{\overline{j}} \\ u_i & u_{i\overline{j}} \end{pmatrix}. \tag{2.26}$$

If u solves the complex Monge-Ampère equation then

$$-\left(\log\left(\frac{1}{u}\right)\right)_{j\overline{k}} dz^j \otimes dz^{\overline{k}}$$

is a Kähler-Einstein metric.

Fefferman [6] showed that there is a smooth function $\psi \in \mathcal{C}^\infty(\Omega)$ that satisfies

$$J[\varphi] = 1 + \mathcal{O}(\varphi^{m+1})$$

$$\varphi|_{\partial\Omega} = 0$$

and that ψ is uniquely determined up to order $m+1$. Cheng and Yau [3] showed the existence of an exact solution belonging to $\mathcal{C}^\infty(\Omega) \cap C^{m+3/2-\epsilon}(\overline{\Omega})$, while Lee and Melrose [23] showed that the exact solution has an asymptotic expansion with logarithmic terms beginning at order $m+2$.

We will show that Fefferman's local approximate solution of the Monge-Ampère equation [6] can be globalized to an approximate solution of the Monge-Ampère equation near the boundary of a complex manifold X. We will see later that, to globalize Fefferman's construction, we need to impose a geometric condition on the CR-structure of M inherited from the complex structure of X. For the convenience of the reader, we review the properties of the operator J under a holomorphic coordinate change and the connection between solutions of the Monge-Ampére equation and Kähler-Einstein metrics.

If $f : \Omega \subset \mathbb{C}^m \to \mathbb{C}^m$ is holomorphic, then f' denotes the matrix

$$(f')_{jk} = \frac{\partial f_j}{\partial z_k}.$$

Lemma 2.5. *Let f be a local biholomorphism. Then, for any local, smooth function u on Ω,*

$$J\left[|\det(f')|^{-2/(m+1)}(u \circ f)\right] = J[u] \circ f.$$

A proof was given by Fefferman in [6]. Here we give an alternative proof using the following identity.

Lemma 2.6. *The formula*

$$J\left[u\right]=u^{m+1}\det\left[\left(\log\left(\frac{1}{u}\right)\right)_{j\overline{k}}\right] \tag{2.27}$$

holds.

Proof. Using row-column operations, one proves that

$$\det\begin{pmatrix} u & u_{\overline{k}} \\ u_j & u_{j\overline{k}}\end{pmatrix}=u\det\left(u_{j\overline{k}}-\frac{u_j u_{\overline{k}}}{u}\right). \tag{2.28}$$

On the other hand, the identity

$$\left(\log\left(\frac{1}{u}\right)\right)_{j\overline{k}}=-\frac{u_{j\overline{k}}}{u}+\frac{u_j u_{\overline{k}}}{u^2}$$

shows that

$$J[u]=(-1)^m u\det\left(u_{j\overline{k}}-\frac{u_j u_{\overline{k}}}{u}\right)=u^{m+1}\det\left(\left(\log\left(\frac{1}{u}\right)\right)_{j\overline{k}}\right). \tag{2.29}$$

Combining (2.28) and (2.29) shows that (2.27) holds. □

We can use the formula (2.27) to show that if u solves the Monge-Ampère equation, then u is the Kähler potential of a Kähler-Einstein metric. Recall that if

$$g=v_{j\overline{k}}\,dz^j\otimes\,dz^{\overline{k}}$$

then the Ricci curvature is

$$R_{a\overline{b}}=-\left[\log\det(v_{j\overline{k}})\right]_{a\overline{b}}$$

Now let

$$v=\log\left(\frac{1}{u}\right)$$

where $J\left[u\right]=1$. Then

$$\begin{aligned}R_{a\overline{b}}&=-\left[\log\det(v_{j\overline{k}})\right]_{a\overline{b}}\\&=-\left[\log\left(u^{-(m+1)}\right)\right]_{a\overline{b}}\\&=-(m+1)\left(\log\left(\frac{1}{u}\right)\right)_{a\overline{b}}\\&=-(m+1)g_{a\overline{b}}\end{aligned}$$

which is the Einstein equation.

Now we prove Lemma 2.5. First, we compute

$$\begin{aligned}\left[\log\left(\left|\det f'\right|^{-2/(m+1)}\ u\circ f\right)\right]_{j\overline{k}}&=\frac{-1}{m+1}\left[\log\left(\left|\det f'\right|^2\right)\right]_{j\overline{k}}\\&\quad+\left[\log\left(u\circ f\right)\right]_{j\overline{k}}\\&=\left[\log\left(u\circ f\right)\right]_{j\overline{k}}\end{aligned} \tag{2.30}$$

where the first right-hand term vanishes because $\left|\det f'\right|^2=(\det f')\,\overline{(\det f')}$ and $\det f'$ is holomorphic. We note that the vanishing of the first term also shows that the Kähler metric with Kähler potential u (when u solves the Monge-Ampère equation) is invariant whether u is considered as a scalar function or a density.

To compute the nonzero right-hand term in (2.30) we first note that if f is a holomorphic map then we have the identity

$$(v \circ f)_{j\bar{k}} = \left[(f')^t \left(v_{a\bar{b}} \circ f \right) \left(\overline{f'} \right) \right]_{j\bar{k}}.$$

Thus, using (2.27), we compute

$$\begin{aligned} J\left[|\det(f')|^{-2/(m+1)}\, u \circ f \right] &= |\det(f')|^{-2} \,(u \circ f)^{m+1} \\ &\quad \times \det\left((f')^t \right) \det\left(\log\left(\frac{1}{u} \right)_{a\bar{b}} \circ f \right) \det\left(\overline{f'} \right) \\ &= (u \circ f)^{m+1} \det\left(\log\left(\frac{1}{u} \right)_{a\bar{b}} \right) \circ f. \\ &= J\left[u \right] \circ f \end{aligned}$$

as was to be proved.

It is essential for our globalization argument that an approximate solution to the Monge-Ampere equation be determined uniquely up to a certain order. This proof was given by Fefferman [6] and we repeat it for the reader's convenience.

Lemma 2.7. *Any smooth, local, approximate solution* $\psi \in \mathcal{C}^\infty(\Omega)$ *to the Monge-Ampère equation is uniquely determined up to order* $m + 1$.

Proof. Suppose that ρ is a smooth function on Ω defined in a neighborhood of $\partial\Omega$ with $\rho = 0$ on $\partial\Omega$ and $\rho'(p) \neq 0$ for all $p \in \partial\Omega$. We recall Fefferman's iterative construction of an approximate solution u to the Monge-Ampère equation, i.e., a function $u \in \mathcal{C}^\infty$ with $u|\partial\Omega = 0$ and $J[u] = 1 + \mathcal{O}(u^{m+2})$. To obtain a first approximation, note that for ρ as above, and for any smooth function η, we have

$$J\left[\eta\rho\right] = \eta^{m+1} J[\rho], \tag{2.31}$$

when $\rho = 0$, so that the function

$$\psi^{(1)} = \rho \cdot J[\rho]^{-1/(m+1)}$$

satisfies $J[\psi^{(1)}] = 1$ on $\partial\Omega$, and $J[\psi^{(1)}] = 1 + \mathcal{O}(\psi^{(1)})$. The fact that $J[\rho]$ is nonzero on $\partial\Omega$ follows from pseudoconvexity that implies that $\rho_{j\bar{k}}$ is positive definite on $\ker\, \partial\rho$ on $\partial\Omega$, and that $\rho' \neq 0$ on $\partial\Omega$. Note that if φ and ψ are two functions vanishing on $\partial\Omega$, it follows that $\varphi = \eta\psi$ for some smooth function η. Thus, by (2.31), $J[\varphi] = \eta^{m+1} J[\psi]$. From this computation it follows that any approximate solution u is uniquely determined up to first order.

We now iterate this construction. Suppose that for an integer $s \geq 2$, we have an approximate solution to the Monge-Ampère equation to order $s - 1$. That is, we have a smooth function ψ with $\psi = 0$ on $\partial\Omega$, $\psi'(p) \neq 0$, for all $p \in \partial\Omega$, and $J[\psi] = 1 + \mathcal{O}(\psi^{s-1})$. We seek a function of the form $v = \psi + \eta\psi^s$, where $\eta \in \mathcal{C}^\infty$ is chosen so that $J[v] = 1 + \mathcal{O}(\psi^s)$. The iteration is based on formula

$$J[\psi + \eta\psi^s] = J[\psi] + \left[s(m + 2 - s) \right] \eta\psi^{s-1} + \mathcal{O}(\psi^s),$$

for smooth functions ψ and η, again with the property that ψ vanishes on $\partial\Omega$. This formula is a straightforward computation using the formula (2.26). From this formula it follows that the desired function v is given by

$$v = \psi + \left[\frac{1 - J(\psi)}{s(m + 2 - s)} \right] \psi^s.$$

The iteration clearly works up to $s = m+1$ and produces an approximate solution with the desired properties. It also follows that any function $\widetilde{u}$ with $u - \widetilde{u} = \mathcal{O}(\psi^{m+2})$ satisfies $J[\widetilde{u}] = J[u] + \mathcal{O}(\psi^{m+2})$. Thus, in particular, any smooth function having the same $(m+1)$-jet on $\partial\Omega$ as an approximate solution is also an approximate solution.

On the other hand, it is clear that any two approximate solutions *must* have the same $(m+1)$-jet on $\partial\Omega$. If ψ and $\widetilde{\psi}$ satisfy $\psi - \widetilde{\psi} = \eta\psi^s$ then $J[\psi] - J[\widetilde{\psi}] = s\,(m+2-s)\,\eta\psi^{s-1} + \mathcal{O}(\psi^s)$. In particular, if $s < m+2$ and $J[\psi] - J[\widetilde{\psi}] = \mathcal{O}(\psi^{m+2})$ then ψ and $\widetilde{\psi}$ are approximate solutions uniquely determined up to order $m+2$.

$\square$

2.4.2. *Global Theory.* Now suppose X is a compact complex manifold of dimension $m = n+1$ with boundary $M = \partial X$. Note that M has real dimension $2n+1$ and inherits an integrable CR-structure from X. As always, we assume that M with this CR-structure is strictly pseudoconvex. We first say what it means for a single smooth function φ defined in a neighborhood of M to be an approximate solution of the complex Monge-Ampère equation. We denote by $\mathcal{C}^\infty(X)$ the smooth functions on X.

Definition 2.8. *We will say that a function $\varphi \in \mathcal{C}^\infty(X)$ is a globally defined approximate solution of the complex Monge-Ampère equation near $M = \partial X$ if for any $p \in M$, there is a neighborhood V of p in X and holomorphic coordinates z on V so that φ is an approximate solution of the complex Monge-Ampère equation in the chosen coordinates.*

As we will see later, we will need such a globally defined approximate solution in order to identify the residues of the scattering operator on X with CR-covariant differential operators.

If φ is a defining function for M with $\varphi < 0$ in the interior of X, we associate to φ a Kähler form

$$\omega_\varphi = \frac{i}{2}\partial\overline{\partial}\log(-1/\varphi) \tag{2.32}$$

and a pseudo-Hermitian structure

$$\theta = \frac{i}{2}\left(\overline{\partial} - \partial\right)\varphi\Big|_M . \tag{2.33}$$

Observe that two defining functions φ and ρ generate the same Kähler metric if and only if $\rho = e^F\varphi$ for a pluriharmonic function F, i.e. $\partial\overline{\partial}F = 0$. It is known that a pluriharmonic function F is uniquely determined by its boundary values (see, for example, Bedford [1]). If θ_ρ and θ_φ are the corresponding pseudo-Hermitian structures on M then $\theta_\rho = e^f\theta_\varphi$, where $f = F|_M$.

We give a necessary and sufficient condition on M for a globally defined approximate solution of the Monge-Ampère equation to exist. Recall that the canonical bundle of M is the bundle generated by forms $f\,\theta^1 \wedge \cdots \wedge \theta^n \wedge \theta$ where f is smooth, θ is a contact form, and $\{\theta^\alpha\}_{\alpha=1}^n$ is an admissible co-frame. If M is the boundary of a strictly pseudoconvex domain in $\mathbb{C}^m$, the canonical bundle is generated by restrictions of forms $f\,dz^1 \wedge \cdots \wedge dz^m$ to M. The sections of the canonical bundle are $(n+1,0)$-forms ζ on M.

If θ is a contact form, T is the characteristic vector field, and ζ is *any* nonvanishing section of the canonical bundle, it is not difficult to see that

$$\theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \overline{\zeta} = \lambda \, \theta \wedge (d\theta)^n$$

for a smooth positive function λ. We say that the contact form θ is *volume-normalized* with respect to a nonvanishing section ζ of the canonical bundle if

$$\theta \wedge (d\theta)^n = (i)^{n^2} \, n! \theta \wedge (T \lrcorner \zeta) \wedge \left(T \lrcorner \overline{\zeta}\right)$$

where T is the characteristic vector field. The following criterion will be useful.

Lemma 2.9. *The contact form θ given by (2.33) is volume-normalized with respect to the form $\zeta = dz^1 \wedge \cdots \wedge dz^m\big|_M$ if and only if*

$$J[\varphi] = 1 + \mathcal{O}(\varphi)$$

in the coordinates $(z_1, \cdots, z_m)$.

For the proof see Farris [5], Proposition 5.2. Using the lemma, we can prove:

Proposition 2.10. *Suppose that X is a compact complex manifold with boundary $M = \partial X$. There is a globally defined approximate solution φ of the Monge-Ampère equation in a neighborhood of M if and only if M admits a pseudo-Hermitian structure θ with the following property: In a neighborhood of any point $p \in M$, there is a local, closed $(n+1, 0)$ form ζ such that θ is volume-normalized with respect to ζ.*

Proof. (i) First, suppose that X admits a globally defined approximate solution φ of the Monge-Ampère equation. Let θ be the associated contact form on X, i.e., θ is given by (2.33). Pick $p \in M$ and let $z \equiv (z_1, \cdots, z_m)$ be holomorphic coordinates near p so chosen that φ is an approximate solution of the Monge-Ampère equation near p in these coordinates. Let

$$\zeta = dz^1 \wedge \cdots \wedge dz^m\big|_M .$$

Then θ is volume-normalized with respect to ζ by Lemma 2.9.

(ii) Suppose that θ is a given contact form on M with the property that, for each point $p \in M$, there is a neighborhood of p and a closed, locally defined section ζ of the canonical bundle with respect to which θ is volume-normalized. Write

$$\zeta = f \quad dz^1 \wedge \cdots \wedge dz^m\big|_M$$

for holomorphic coordinates $\{z_1, \cdots, z_m\}$ defined in a neighborhood of p and a smooth function f. The condition $d\zeta = 0$ is equivalent to the condition

$$\overline{\partial_b} f = 0$$

i.e., f is a CR-holomorphic function. By the strict pseudoconvexity of M, there is a holomorphic extension F to a neighborhood V of p in X, i.e., there is an F defined near p with $\overline{\partial} F = 0$ and $F|_{M \cap V} = f$ (see [20]). We claim that we can find new holomorphic coordinates $w \equiv (w_1, \cdots, w_m)$ near p with the property that

$$\frac{\partial \left(w_1, \cdots, w_m\right)}{\partial \left(z_1, \cdots, z_m\right)} = F(z) \tag{2.34}$$

If so then

$$\zeta = dw^1 \wedge \cdots \wedge dw^m\big|_M$$

Constructing in V an approximate solution ψ_V of the Monge-Ampère equation in the w-coordinates (as in Lemma 2.7, following Fefferman [6]), we conclude from Lemma 2.9 that the induced contact form

$$\theta_V = \frac{i}{2}\left(\overline{\partial} - \partial\right)\psi_V \Big|_{M\cap V}$$

on $M \cap V$ is volume-normalized with respect to ζ, and thus coincides with θ.

We now claim that the local approximate solutions ψ_V can be glued together to form a globally defined approximate solution to the Monge-Ampère equation in the sense of Definition 2.8. We first note an important property of the transition map for two local coordinates. Let V_1 and V_2 be neighborhoods of M in X with nonempty intersection, let z and w be holomorphic coordinates on V_1 and V_2, and suppose that ψ_1 and ψ_2 are approximate solutions of the complex Monge-Ampère equation in these respective coordinates. More precisely, $u_1 = \psi_1 \circ z$ and $u_2 = \psi_2 \circ w$ are approximate solutions to the Monge-Ampère equation on coordinate patches U_1 and U_2 in $\mathbb{C}^m$, and there is a biholomorphic map $g : U_2 \cap w^{-1}(V_1 \cap V_2) \rightarrow U_1 \cap z^{-1}(V_1 \cap V_2)$. The function $u_2 = |g'|^{2/(m+1)}\, u_1 \circ g$ is also an approximate solution of the complex Monge-Ampere equation in $U_2 \cap w^{-1}(V_1 \cap V_2)$ by Lemma 2.5, so by uniqueness we have $u_2 = e^F u_1 \circ g$, up to order $m+1$, where $F = (2/(m+1)) \log |g'|$ is pluriharmonic. Moreover, since u_1 and u_2 both induce the contact form θ it follows that

$$\left(\overline{\partial} - \partial\right) u_2\big|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)} = \left[\left(\overline{\partial} - \partial\right) u_1\right] \circ f\big|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)}$$

from which we deduce that $F|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)} = 0$, and hence $F = 0$ by the uniqueness of pluriharmonic extensions. In particular, the map g is unimodular, $|g'| = 1$. Thus $u_2 = u_1 \circ g$ on $U_2 \cap w^{-1}(V_1 \cap V_2)$ up to order $m+1$.

We now fix a boundary defining function ρ. Suppose that $\{U_i\}$ is a finite cover of a neighborhood of the boundary by holomorphic charts. Denote by F_i the map from $\mathbb{C}^m$ into U_i and set $F_{ij} = F_i^{-1} \circ F_j$. As proved above, the cover and holomorphic coordinates (U_i, F_i) may be chosen so that the transition maps are unimodular, i.e., $\left|F'_{ij}\right| = 1$. Using Fefferman's construction, we can produce in each U_i an approximate solution u_i in the sense that

$$J\left[u_i\right] = 1 + \mathcal{O}(\rho^{m+1})$$

Now suppose that $\{\chi_i\}$ is a $\mathcal{C}^\infty$ partition of unity subordinate to the cover $\{U_i\}$. We claim that the smooth function $u = \sum_i \chi_i u_i$ is an approximate solution of the Monge-Ampère equation in the sense of Definition 2.8. Choose U_i so that $p \in U$. We may write $u = \sum_j (\chi_j \circ F_i)(z)\ \ (u_j \circ F_i)$. Since $u_j \circ F_i = (u_j \circ F_j) \circ F_{ji}$ we see that $(u_j \circ F_i)$ is also an approximate solution to the Monge-Ampère equation in the F_i-coordinates. Thus, there is a smooth function η_{ji} so that

$$\left(u_j \circ F_i\right)(z) - \left(u_i \circ F_i\right)(z) = \eta_{ji}(z)\left(\rho \circ F_i\right)^{m+2}(z)$$

where η_{ji} is smooth. We conclude that

$$u(z) - u_i(z) = \mathcal{O}((\rho \circ F_i)^{m+2}).$$

This shows that u is also an approximate solution of the Monge-Ampère equation in the F_i-coordinates as claimed.

To finish the proof it suffices to establish that such a holomorphic coordinate change $z \mapsto w$, as in (2.34), exists. We consider a coordinate transformation given

by

$$w(z) = (h(z), z_2, \ldots, z_m), \tag{2.35}$$

where $h(z)$ is the unknown holomorphic function. Condition (2.34) is equivalent to

$$\frac{\partial h}{\partial z_1}(z_1, \ldots, z_m) = F(z_1, z_2, \ldots, z_m). \tag{2.36}$$

Here, F is the holomorphic extension of the CR-function f. We solve this equation for h as follows. We set the convention that a boundary chart in $\mathbb{C}^m$ is the intersection of an open ball about 0 with the (real) half-space $\operatorname{Im} z_m \geq 0$. We assume that the boundary point p corresponds to $0 \in \partial\mathbb{C}^m$. The unknown function h is a complex-valued function defined in a neighborhood V of $0 \in \mathbb{C}^m$, is holomorphic in $V \cap \{\operatorname{Im} z_m > 0\}$, has CR boundary values, and satisfies $h(0) = 0$. Thus, the map $w(z)$, defined in (2.35), preserves the boundary $\operatorname{Im}(z_n) = 0$.

Consequently, the desired change of coordinates is obtained by solving the initial value problem

$$\begin{aligned} &\frac{\partial h}{\partial z_1}(z_1, \cdots, z_m) = F(z_1, z_2, \cdots, z_m) \\ &h(0, z_2, \cdots, z_m) = 0, \end{aligned} \tag{2.37}$$

by simple integration. □

We can also express the basic criterion in Proposition 2.10 in geometric terms. Recall that the contact form θ defines a pseudo-Hermitian, pseudo-Einstein structure on M if the Webster Ricci tensor is a multiple of the Levi form. Lee [22] proved:

Theorem 2.11. *Suppose that M is a CR-manifold of dimension ≥ 5. A contact form θ on M is pseudo-Einstein if and only if for each $p \in M$ there is a neighborhood of p in M and a locally defined closed section ζ of the canonical bundle with respect to which θ is volume-normalized.*

As an immediate consequence of Theorem 2.11, we have:

Theorem 2.12. *Suppose that M is a CR-manifold of dimension ≥ 5. There is a globally defined approximate solution φ of the complex Monge-Ampère equation in a neighborhood of M if and only if M carries a contact form θ for which the corresponding pseudo-Hermitian structure is pseudo-Einstein. In this case, the contact form θ is induced by the globally defined approximate solution to the Monge-Ampère equation φ.*

Remark 2.13. *If φ is a global approximate solution to the Monge-Ampère equation, then so is $e^F\varphi$ where F is any pluriharmonic function. The effect of the factor F is simply to change the choice of local coordinates needed to obtain a local approximate solution of the Monge-Ampère equation in any chart, as the argument in the proof of Proposition 2.10 easily shows. As observed above, the Kähler form ω_φ is invariant under the change $\varphi \mapsto e^F\varphi$.*

3. Poisson Operator and Scattering Operator

In this section we study the Dirichlet problem (1.3) following a standard technique in geometric scattering theory (see, for example, Melrose [24]; we follow closely the analysis of the Poisson operator and scattering operator on conformally

compact manifolds by Graham and Zworski in [14]). Note that Epstein, Melrose, and Mendoza [4] had previously studied the Poisson operator for a class of manifolds that includes compact complex manifolds with strictly pseudoconvex boundaries. More recently, Guillarmou and Sá Barreto [15] studied scattering theory and radiation fields for asymptotically complex hyperbolic manifolds, a class which also includes that studied here.

We will set $x = -\varphi$ and we will denote by $\mathcal{C}^\infty(X)$ the set of smooth functions on X having Taylor series to all orders at $x = 0$, and by $\dot{\mathcal{C}}^\infty(X)$ the space of functions vanishing to all orders at $x = 0$. The space $\mathcal{C}^\infty(\mathring{X})$ consists of smooth functions on $\mathring{X}$ with no restriction on boundary behavior. We will denote by $x^s\mathcal{C}^\infty(X)$ the set of functions in $\mathcal{C}^\infty(\mathring{X})$ having the form x^sF for $F \in \mathcal{C}^\infty(X)$.

Since

$$N = -2\frac{\partial}{\partial x}$$

it follows that

$$\nu = -\frac{x}{\sqrt{1+rx}}\frac{\partial}{\partial x} \tag{3.1}$$

is the outward normal to the hypersurface $x = \varepsilon$. Green's theorem implies that

$$\int_{x>\varepsilon} (u_1\Delta_\varphi u_2 - u_2\Delta_\varphi u_1)\ \omega^m = \int_{x=\varepsilon} (u_1\nu u_2 - u_2\nu u_1)\ \nu \lrcorner\, \omega^m \tag{3.2}$$

We first note the 'boundary pairing formula' (recall the definition (2.16)).

Proposition 3.1. *Suppose* $\mathrm{Re}(s) = m/2$, *that* u_1 *and* u_2 *belong to* $\mathcal{C}^\infty(\mathring{X})$ *and there are functions* $F_i, G_i \in \mathcal{C}^\infty(X)$ *so that* $u_i = x^{m-s}F_i + x^sG_i$, $i = 1, 2$. *Finally, suppose that* $(\Delta_\varphi - s(m-s))\,u_i = r_i \in \dot{\mathcal{C}}^\infty(X)$, $i = 1, 2$. *Then, the formula*

$$\int_X (u_1r_2 - u_2r_1)\ \omega^m = (2s-m)\int_M (F_1G_2 - F_2G_1)\ \psi \tag{3.3}$$

holds.

Proof. A standard computation using (3.2) and (3.1) together with (2.15) and (2.21). ☐

Remark 3.2. *For* $\mathrm{Re}(s) = m/2$ *complex conjugation reverses the roles of* s *and* $m - s$. *Thus we obtain the formula*

$$\int_X (u_1\overline{r_2} - \overline{u_2}r_1)\ \omega^m = (2s-m)\int_M \left(F_1\overline{F_2} - G_1\overline{G_2}\right)\ \psi \tag{3.4}$$

For later use, we note an extension of the boundary pairing formula analogous to Proposition 3.3 of [14].

Proposition 3.3. *Suppose that* $\mathrm{Re}(s) > m/2$ *and* $2s - m \notin \mathbb{N}$. *Suppose that* $u_i \in \mathcal{C}^\infty(\mathring{X})$ *takes the form*

$$u_i = x^{m-s}F_i + x^sG_i \tag{3.5}$$

(3.5) and $(\Delta_\varphi - s(m-s))\,u_i = 0$, $i = 1, 2$. *Then*

$$\operatorname*{FP}_{\varepsilon\downarrow 0}\left(\int_{x>\varepsilon} [\langle\nabla u_1, \nabla u_2\rangle - s(m-s)u_1u_2]\ \omega^m\right) = -m\int_M G_1F_2\ \psi = -m\int_M F_1G_2\ \psi$$

where FP *denotes the Hadamard finite part of the integral as* $\varepsilon \downarrow 0$.

Proof. Green's formula (3.2) for the operator $\Delta_\varphi - s(m-s)$ gives

$$\int_{x>\varepsilon} \left[\langle \nabla u_1, \nabla u_2\rangle - s(m-s)u_1 u_2\right]\omega^m = \int_{x=\varepsilon} u_1\left(\nu u_2\right)\ \nu \lrcorner\, \omega^m$$

from which the claimed formulae follow. □

3.1. **The Poisson Map.** We now prove that the Dirichlet problem (1.3) has a unique solution if $\mathrm{Re}(s) \geq m/2$, $2s - m \notin \mathbb{Z}$, and $s(m-s)$ is not an eigenvalue of Δ_φ. Most of the formal arguments are almost identical to the case of even asymptotically hyperbolic manifolds considered in [14] since the form of the normal operator (2.24) for the Laplacian is the same.

Lemma 3.4. *Suppose that $u \in \mathcal{C}^\infty(\mathring{X})$ satisfies $u = x^{m-s}F + x^s G$ for functions F and G belonging to $\mathcal{C}^\infty(X)$, and that*

$$(\Delta_\varphi - s(m-s))\, u \in \dot{\mathcal{C}}^\infty(X) \tag{3.6}$$

for $s \in \mathbb{C}$ with $2s-m \notin \mathbb{Z}$. Then the Taylor expansions of F and G at $x = 0$ are formally determined respectively by $F|_M$ and $G|_M$. In particular, we have $F \sim \sum_{k\geq 0} x^k f_k$ and $G \sim \sum_{k\geq 0} x^k g_k$ where

$$f_k = \frac{1}{k!}\frac{\Gamma(2s-m-k)}{\Gamma(2s-m)} P_{k,s} f_0 \tag{3.7}$$

and

$$g_k = \frac{1}{k!}\frac{\Gamma(m-2s-k)}{\Gamma(m-2s)} P_{k,m-s} g_0$$

where $P_{k,s}$ are differential operators of order $2k$ holomorphic in s with leading symbol[2]

$$\sigma(P_{k,s}) = \frac{1}{2^k}\sigma\left(-\Delta_b^k\right)$$

Proof. Recall the asymptotic development (2.23) for the Laplacian which we use to derive a recurrence for the Taylor coefficients f_k and g_k of F and G. For $2s-m \notin \mathbb{Z}$, we may consider the terms involving F and G separately. We first consider F. Observe that

$$\left(L_0 - s(m-s)\right)\left(x^{m-s+k} f\right) = k(2s-m-k)x^{s+k} f$$

for $f \in \mathcal{C}^\infty(M)$. Since $L_k = P(x\partial_x, \partial_y)$ for a defining function x and boundary coordinates y where P is a polynomial of degree at most two with smooth coefficients, the operators

$$Q_{k,\ell}(s) = x^{-m+s-\ell} L_{k-\ell} x^{m-s+\ell}$$

are differential operators of order at most two depending holomorphically on s. If $u \sim \sum_{k=0}^\infty x^{m-s+k} f_k$, it follows from (3.6) and (2.23) that for any $k \geq 1$,

$$f_k = -\frac{1}{k(2s-m-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(s) f_\ell \tag{3.8}$$

Similarly, if $u \sim \sum_{k\geq 0} x^{s+k} g_k$ for $g_k \in \mathcal{C}^\infty(M)$, we have

$$g_k = -\frac{1}{k(m-2s-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(m-s) g_\ell \tag{3.9}$$

[2]Here in the sense of the ordinary (rather than the Heisenberg) calculus on M.

The formulas for f_k, g_k, and $P_{k,s}$ follow easily from these formulas and the fact that

$$Q_{k,k-1}(s) = \frac{1}{4}\left(-2\Delta_b u - 4r_0\,(m-s+1) - 4r_0\left[(m-s+1)^2 - (m-s+1)\right]\right)$$

□

Remark 3.5. *We will write $p_{k,s}$ for the operator with $f_k = p_{k,s} f_0$, so that $p_{k,s}$ is meromorphic with poles at $s = m/2 + k/2, \ldots, m/2 + 1/2$. We will denote*

$$p_\ell = \operatorname*{Res}_{s=m/2+\ell/2} p_{\ell,s}$$

The operator p_ℓ is a differential operator of order at most 2ℓ with principal symbol

$$\sigma(p_\ell) = \frac{1}{2^\ell \ell!(\ell-1)!}\sigma\left(-\Delta_b^\ell\right)$$

For $\mathrm{Re}(s) > m/2$, let

$$R(s) = \left(\Delta_\varphi - s(m-s)\right)^{-1}$$

be the $L^2(X)$ resolvent, let $\sigma_p(\Delta_\varphi)$ denote the set of L^2-eigenvalues of Δ_φ, and let

$$\Sigma = \left\{s : \mathrm{Re}(s) > m/2,\ s(m-s) \in \sigma_p(\Delta_\varphi)\right\}.$$

We will now solve the Dirichlet problem (1.3) for $\mathrm{Re}(s) \geq m/2$ and $s \notin \Sigma$.

The following result is an easy consequence of the work of Epstein, Melrose, and Mendoza [4], noting that in our case the Kähler metric is an even metric, i.e., depends smoothly on the defining function φ (and not simply on its square root).

Proposition 3.6. *The set Σ contains at most finitely many points, and the resolvent operator $R(s)$ is a meromorphic operator-valued function for $\mathrm{Re}(s) > m/2-1/2$ having at most finitely many, finite-rank poles at $s \in \Sigma$. Moreover, for $s \notin \Sigma$,and $\mathrm{Re}(s) > m/2 - 1/2$, $R(s) : \dot{\mathcal{C}}^\infty(X) \to x^s \mathcal{C}^\infty(X)$.*

First, we prove uniqueness of solutions to the Dirichlet problem (1.3) for s with $\mathrm{Re}(s) \geq m/2$, $s \notin \Sigma$, and $2s - m \notin \mathbb{Z}$.

Proposition 3.7. *Suppose that $\mathrm{Re}(s) \geq m/2$, $s \notin \Sigma$, and $2s - m \notin \mathbb{Z}$. Suppose that $u \in \mathcal{C}^\infty(\mathring{X})$ with $\left(\Delta_\varphi - s(m-s)\right)u = 0$, and that $u = x^{m-s}F + x^s G$ with $F|_M = 0$. Then $u = 0$.*

Proof. First, suppose that $\mathrm{Re}(s) > m/2$ and $s \notin \Sigma$. It follows from Lemma 3.4 that $u = x^s G$ for $G \in \mathcal{C}^\infty(X)$. Since $\mathrm{Re}(s) > m/2$ it is clear that $\int_X |u|^2\ \omega^m < \infty$, hence $u \in L^2(X)$, hence $u = 0$.

If $\mathrm{Re}(s) = m/2$ but $s \neq m/2$, we may again assume that $u = x^s G$ for $G \in \mathcal{C}^\infty(X)$. Next, we set $u_1 = u_2 = u$ in (3.4) to conclude that $\int_M |G|^2\,\psi = 0$ so that $G|_M = 0$. Using Lemma 3.4 again we conclude that $G \in \dot{\mathcal{C}}^\infty(X)$, hence $u \in \dot{\mathcal{C}}^\infty(X)$. As in [15], we can now deduce from [28] that $u = 0$. □

To prove existence of a solution of the Dirichlet problem (1.3), we follow the method of Graham and Zworski [14]. Given $f \in \mathcal{C}^\infty(M)$ we can construct a formal power series solution $u = x^{n-s}F$ modulo $\dot{\mathcal{C}}^\infty(X)$, and then use the resolvent to correct this approximate solution to an exact solution. Using Borel's lemma we can sum the asymptotic series $\sum_{j\geq 0} f_j x^j$ (where f_j is computed via (3.8) with $f_0 = f$) to a function $F \in \mathcal{C}^\infty(X)$. As in [14], we obtain:

Lemma 3.8. *There is an operator* $\Phi(s) : \mathcal{C}^\infty(M) \to x^{n-s}\mathcal{C}^\infty(X)$ *with*

$$(\Delta_\varphi - s(m-s)) \circ \Phi : \mathcal{C}^\infty(M) \to \dot{\mathcal{C}}^\infty(X)$$

so that $\Gamma(m-2s)^{-1}\Phi(s)$ *is holomorphic in* s.

Note that $\Phi(s)$ need not be linear as the construction of F depends on the choice of cutoff functions in the application of Borel's lemma. Now define an operator

$$\mathcal{P}(s) : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\mathring{X})$$

for s with $\operatorname{Re}(s) \geq m/2$, $s \neq m/2$ and $s \notin \Sigma$ by

$$\mathcal{P}(s) = [I - R(s)\,(\Delta_\varphi - s(m-s))] \circ \Phi(s)$$

Lemma 3.9. *For any* $f \in \mathcal{C}^\infty(M)$*, the function* $u = \mathcal{P}(s)f$ *solves the Dirichlet problem (1.3), and* $f \mapsto \mathcal{P}(s)f$ *is a linear operator.*

Proof. The linearity of $\mathcal{P}(s)$ will follow from the unicity of the solution to (1.3). It is immediate from the definition that $(\Delta_\varphi - s(m-s))\,u = 0$, and from the mapping property in Proposition 3.6, $u = x^{m-s}F + x^sG$ with $F = x^{s-m}\Phi(s)f$ and $G = -x^{-s}R(s)\,[(\Delta_\varphi - s(m-s))\,\Phi(s)f]$. □

We now have:

Theorem 3.10. *For* $\operatorname{Re}(s) \geq m/2$*,* $2s - m \notin \mathbb{Z}$*, and* $s \notin \Sigma$*, there exists a unique solution of the Dirichlet problem (1.3).*

3.2. **The Scattering Operator.** The *scattering operator* for Δ_φ is the linear mapping

$$\begin{aligned} S_X(s) : \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M) \\ f &\mapsto G|_M \end{aligned}$$

where $u = x^{m-s}F + x^sG$ solves (1.3). It is well-defined by Theorem 3.10.

The scattering operator has infinite-rank poles when $\operatorname{Re}(s) > m/2$ and $2s - m \in \mathbb{Z}$ owing to the crossing of indicial roots for the normal operator L_0. At the exceptional points $s = m/2 + k$ one expects solutions of the eigenvalue equation $(\Delta_\varphi - s(m-s))\,u = 0$ having the form

$$u = x^{m/2-k}F + \left(x^{m/2+k}\log x\right)G$$

In order to study the singularities of the scattering operator at these points we modify the construction of the Poisson operator following the lines of [14], section 4.

Let f_1 and f_2 belong to $\mathcal{C}^\infty(M)$ and let u_1 and u_2 solve the corresponding Dirichlet problems for some s with $\operatorname{Re}(s) > m/2$ and $2s - m \notin \mathbb{N}$. Applying the generalized boundary pairing formula (see Proposition 3.3) to u_1 and $\overline{u_2}$ for s real, we conclude that

$$\int_M f_1 \overline{S_X(s) f_2}\; \psi = \int_M [S_X(s) f_1]\, \overline{f_2}\; \psi$$

so $S_X(s)$ is self-adjoint in the natural inner product on $\mathcal{C}^\infty(M)$.

Now we study the scattering operator near the exceptional points. The arguments used here are exactly those of section 3 in [14] but we summarize them here for the reader's convenience.

Recall the operators $p_{k,s}$ and p_ℓ defined in Remark 3.5. First, we prove:

Lemma 3.11. *At the points $s = m/2 + \ell/2$, $\ell = 1, 2, \cdots$, $s \notin \Sigma$, the Poisson map takes the form*

$$\mathcal{P}(m/2 + \ell/2)f = x^{m/2-\ell/2}F + \left(x^{n/2+\ell/2}\log x\right)G$$

where

$$F|_M = f$$

and

$$G|_M = -2p_\ell f$$

where

$$p_\ell = \operatorname*{Res}_{s=m/2+\ell/2} p_{\ell,s} \tag{3.10}$$

is a differential operator of order 2ℓ with

$$\sigma(p_\ell) = \frac{1}{2^\ell \ell!\,(\ell-1)!}\sigma(\Delta_b^\ell)$$

Proof. We first show that the Poisson map $\mathcal{P}(s)$ is also regular at $s = m/2 + \ell/2$, $\ell = 1, 2, \cdots$ so long as these points do not belong to Σ. As in [14] we introduce the operator

$$\Phi_\ell(s) = \Phi(s) - \Phi(m-s) \circ p_{\ell,s} \tag{3.11}$$

where $p_{\ell,s}$ is a differential operator of order 2ℓ defined in Remark 3.5. Each of the right-hand terms in (3.11) has at most a first-order pole at $s = m/2 + \ell/2$; the operators $p_{j.s}$ occurring in the definition of $\Phi(s)$ have at most first-order poles, while $\Phi(m-s)$ is analytic in s for $\mathrm{Re}(s) > m/2$. For given $f \in \mathcal{C}^\infty(M)$, we compute the residue of $\Phi_\ell(s)f$ at $s = m/2 + \ell/2$. First

$$\lim_{s\to m/2+\ell/2}\left(s - \frac{m}{2} - \frac{\ell}{2}\right)\Phi(s)f = x^{m/2+\ell/2}\operatorname*{Res}_{s=m/2+\ell/2}(p_{\ell,s}f) + \mathcal{O}\left(x^{m/2+\ell/2+1}\right)$$

since the remaining terms in the asymptotic expansion for $\Phi(s)f$ are holomorphic near $s = m/2 + \ell/2$. Second,

$$\lim_{s\to m/2+\ell/2}\left(s - \frac{m}{2} - \frac{\ell}{2}\right)\Phi(m-s)\,(p_{\ell,s}f) = x^{m/2+\ell/2}\operatorname*{Res}_{s=m/2+\ell/2}(p_{\ell,s}f) + \mathcal{O}\left(x^{m/2+\ell/2+1}\right).$$

It follows that

$$\operatorname*{Res}_{s=m/2+\ell/2}\Phi_\ell(s)f = \mathcal{O}\left(x^{m/2+\ell/2+1}\right) \tag{3.12}$$

so that, by Lemma 3.4, $\mathrm{Res}_{s=m/2+\ell/2}\,\Phi_\ell(s)f \in \dot{\mathcal{C}}^\infty(X)$.

Now let us define

$$\mathcal{P}_\ell(s) = [I - R(s)\,(\Delta_\varphi - s(m-s))] \circ \Phi_\ell(s)$$

Clearly, $\mathcal{P}_\ell(s)$ is holomorphic in a deleted neighborhood of $s = m/2 + \ell/2$ (with at most a first-order pole at $s = m/2 + \ell/2$) and maps $\mathcal{C}^\infty(M)$ into $\mathcal{C}^\infty(\mathring{X})$. If $s \notin \Sigma$, it follows from the definition of $\mathcal{P}_\ell(s)$, equation (3.12), and Proposition 3.6 that

$$\operatorname*{Res}_{s=m/2+\ell/2}\mathcal{P}_\ell(s)f \in x^s\mathcal{C}^\infty(X),$$

hence the residue is an $L^2(X)$ function, and hence is zero. Thus $\mathcal{P}_\ell(s)$ is holomorphic at $s = m/2 + \ell/2$. It follows from the uniqueness of solutions to the Dirichlet

problem that $\mathcal{P}_\ell(s) = \mathcal{P}(s)$ wherever the former is defined. Exactly as in [14] we can compute $\mathcal{P}(m/2+\ell/2)f$ by using $\mathcal{P}_\ell(s)$, the formula

$$\lim_{t\to 0} \frac{x^{-t}-x^t}{t} = -2\log x$$

and the fact that the $p_{k,s}$ have at most simple poles at $s = m/2+\ell/2$. This computation shows that $\mathcal{P}(m/2+\ell/2)$ has the stated form. □

Next, we prove:

Proposition 3.12. *Suppose that Δ_X has no eigenvalues of the form $s(m-s)$ with $s = m/2+\ell/2$, $\ell = 1,2,\cdots$. Then, the scattering operator $S_X(s)$ has a first-order pole at $s = m/2+\ell/2$, $\ell = 1,2,\cdots$ with*

$$\operatorname*{Res}_{s=m/2+\ell/2} S_X(s) = p_\ell.$$

where p_ℓ is the differential operator given by (3.10).

Proof. From the formula for the $\mathcal{P}_\ell(s)$, it is clear that for $2s-m \notin \mathbb{N}$, we can compute the scattering operator from

$$S_X(s)f = \left[-x^{-s}R(s)\left(\Delta_\varphi - s(m-s)\right)\Phi(s)f\right]\Big|_{x=0}.$$

Since $\mathcal{P}(s)$ is holomorphic at $s = n/2+\ell/2$ (unless $s\in\Sigma$), it follows that

$$\operatorname*{Res}_{s=m/2+\ell/2}\left[S_X(s)f\right] = \operatorname*{Res}_{s=m/2+\ell/2}\left[x^{-s}\Phi(s)f\big|_{x=0}\right].$$

But

$$\begin{aligned}\operatorname*{Res}_{s=m/2+\ell/2}\left(x^{-s}\Phi(s)f\big|\right)_{x=0} &= \operatorname*{Res}_{s=m/2+\ell/2}\left(\left[x^{-s}\Phi(m-s)p_{\ell,s}f\right]\big|_{x=0}\right)\\ &= \operatorname*{Res}_{s=m/2+\ell/2}\left[p_{\ell,s}f\right]\end{aligned}$$

and the claimed formula holds. □

To connect the scattering operator and the CR Q-curvature, we will also need the following result about the pole of the scattering operator at $s = m$; this result is a direct analogue of Proposition 3.7 in [14] but we give the short proof for the reader's convenience.

Proposition 3.13. *Let 1 denote the constant function on M. Then, the formula*

$$S_X(m)1 = -\lim_{s\to m} p_{m,s}(1)$$

holds.

Proof. As $s \to m$ we have $\mathcal{P}(s)1 \to 1$. On the other hand, for s with $|s-m| < 1/2$,

$$\mathcal{P}(s)1 = \sum_{k=0}^{m} x^{m-s+k}p_{k,s}(1) + x^s S_X(s)1 + \mathcal{O}(x^{m+1/2}).$$

This implies that

$$\lim_{s\to m}\left[x^{2m-s}p_{m,s}(1) + x^m S_X(s)1\right] = 0$$

from which the claimed formula follows. □

Remark 3.14. *Note that, although $p_{m,s}$ has a pole at $s = m$, the limit $\lim_{s \to m} p_{m,s}(1)$ exists. This implies that $P_{m,s}1$ (see (3.7)) has a first-order zero at $s = m$, i.e., $P_{m,s}1 = (m - s)Q_{m,s}$ for a scalar function $Q_{m,s}$. The CR Q-curvature is then given by $Q_{m,m}$* [8].

4. CR-Covariant Operators

In this section we show that if φ is an approximate solution of the complex Monge-Ampère equation in the sense discussed above, then the residues of the scattering operator at $s = m/2 + \ell/2$, $\ell = 1, \cdots, m$ are the CR-covariant differential operators P_k defined in [8]. In order to do this we first recall Fefferman and Graham's [7] set-up for studying conformal invariants of compact manifolds and the construction of the GJMS [12] operators. We then recall its application to CR-manifolds taking care that the arguments carry over from pseudoconvex domains in $\mathbb{C}^m$ to the manifold setting studied here.

4.1. The GJMS Construction.

We begin by recalling Fefferman and Graham's construction of the ambient metric and ambient space for a conformal manifold and the GJMS conformally covariant operators on $\mathcal{C}$ obtained from this construction. Suppose that $(\mathcal{C}, [g])$ is a conformal manifold of signature (p, q), i.e., a smooth manifold of dimension $N = p + q$ together with a conformal class of pseudo-Riemannian metrics of signature (p, q) on $\mathcal{C}$. Fix a conformal representative g_0. The *metric bundle* $\mathcal{G} \subset S^2T^*\mathcal{C}$ is a bundle on $\mathcal{C}$ with fibres

$$\mathcal{G}_p = \left\{t^2 g_0(p) : t > 0\right\}$$

We denote by $\pi : \mathcal{G} \to M$ the natural projection. The *tautological metric* G on $\mathcal{G}$ is given by

$$G(X, Y) = g(\pi_* X, \pi_* Y)$$

for tangent vectors X and Y to $(p, g) \in \mathcal{G}$. There is a natural $\mathbb{R}^+$-action δ_s on $\mathcal{G}$ given by $\delta_s(p, g) = (p, s^2 g)$.

The *ambient space* over $\mathcal{C}$ is the space $\widetilde{\mathcal{G}} = \mathcal{G} \times (-1, 1)$. Note that the map $g \mapsto (g, 0)$ imbeds $\mathcal{G}$ in $\widetilde{\mathcal{G}}$.

Fefferman and Graham proved the existence of a unique metric $\widetilde{g}$ of signature $(p + 1, q + 1)$ on $\widetilde{\mathcal{G}}$, the *ambient metric* on $\widetilde{\mathcal{G}}$ having the following three properties:
(a) $i^*\widetilde{g} = G$
(b) $\delta_s^*\widetilde{g} = s^2\widetilde{g}$
(c) $\mathrm{Ric}(\widetilde{g}) = 0$ along $\mathcal{G}$ to infinite order if N is odd, and up to order $N/2$ if N is even.

Here the uniqueness is meant in the sense of formal power series.

To define the GJMS operators, we first define spaces of homogeneous functions on $\mathcal{G}$. For $w \in \mathbb{R}$ let $\mathcal{E}(w)$ denote the functions f on $\mathcal{G}$ homogeneous of degree w with respect to δ_s and smooth away from 0. The GJMS operators $\mathcal{P}_k$ may be defined in two ways.

(1) Given $f \in \mathcal{E}(-N/2 + k)$, extend f to a function $\widetilde{f}$ homogeneous of the same degree on $\widetilde{\mathcal{G}}$, and set

$$\mathcal{P}_k f = \widetilde{\Delta}^k \widetilde{f}\Big|_{\mathcal{G}} \tag{4.1}$$

where $\widetilde{\Delta}$ is the Laplacian for the ambient metric $\widetilde{g}$ on $\widetilde{\mathcal{G}}$.

(2) Given $f \in \mathcal{E}(-N/2+k)$, $\mathcal{P}_k$ is the normalized obstruction to extending f to a smooth function $\widetilde{f}$ on $\widetilde{\mathcal{G}}$ having the same homogeneity and satisfying $\widetilde{\Delta}^k \widetilde{f} = 0$.

The existence of GJMS operators was proven in [12] for $k = 1, 2, \cdots$ if N is odd, and for $k = 1, 2, \cdots, N/2$ if N is even.

4.2. **Application to CR-Manifolds.** Following [9] we describe how the GJMS construction [12] can be used to prove the existence of CR-covariant differential operators. We begin with a CR-manifold M of dimension $2n+1$ and show how to construct a conformal manifold $\mathcal{C}$ of dimension $2n+2$ and a conformal class of metrics with signature $(2n+1, 1)$ to which the GJMS construction may be applied. One then "pulls back" the GJMS operators to M.

Recall that the *canonical bundle* K over M is the bundle of holomorphic $(n+1)$-forms generated by holomorphic forms of the type $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$ where θ is a contact form and $\{\theta^\alpha\}$ is a basis for $\mathcal{H}$ of admissible $(1,0)$-forms. We denote by K^* the canonical bundle of M with the zero section removed. The *circle bundle* $\mathcal{C}$ over M is the bundle

$$\mathcal{C} = (K^*)^{1/(n+2)} / \mathbb{R}^+.$$

The circle bundle is an S^1-bundle over M, having real dimension $2m$ if $m = n+1$. If we fix a contact form θ on M (and hence a pseudo-Hermitian structure on M), there is a corresponding section ζ of K^* chosen so that θ is volume-normalized with respect to ζ. We denote by ψ the angle determined by $\zeta(p)$ in each fibre of $\mathcal{C}$ and define a fibre variable

$$\gamma = \frac{\psi}{n+2}$$

Note that γ is canonically determined by θ. Following Lee [21], let us define a canonical one-form σ on $\mathcal{C}$ by

$$(n+2)\sigma = (n+2)d\gamma + i\omega_\alpha^{\ \alpha} - \frac{1}{2(n+1)} R\theta \tag{4.2}$$

where $\omega_\alpha^{\ \beta}$ is the connection one-form and R is the Webster scalar curvature of the pseudo-Hermitian structure θ. The mapping $\theta \mapsto g_\theta$ given by

$$g_\theta = h_{\alpha\beta}\theta^\alpha \cdot \theta^{\overline{\beta}} + 2\theta \cdot \sigma \tag{4.3}$$

(where $\cdot$ denotes the symmetric product) defines a mapping of pseudo-Hermitian structures to Lorenz metrics which respects conformal classes. One can now obtain GJMS operators on $\mathcal{C}$ using the Fefferman-Graham construction.

Remark 4.1. *It is immediate from formulas (4.2) and (4.3) that*

$$g_\theta(T,T) = -\frac{1}{(n+1)(n+2)} R.$$

On the other hand, Farris [5] *computed that, if θ is the contact form induced by an approximate solution of the complex Monge-Ampère equation, then*

$$g_\theta(T,T) = 2r$$

where r is the transverse curvature. It follows that the transverse curvature is, in this case, an intrinsic pseudo-Hermitian invariant.

To compute their pullbacks to M, we first note that the metric bundle $\mathcal{G}$ of $(\mathcal{C},[g])$ is diffeomorphic to $(K^*)^{1/(n+2)}$ and $\widetilde{\mathcal{G}} \simeq (K^*)^{1/(n+1)} \times (-1,1)$. We define spaces of functions

$$\begin{aligned}\mathcal{E}(w,w') &= \left\{ f \in \mathcal{C}^\infty((K^*)^{1/(n+2)} : f(\lambda\xi) = \lambda^w \overline{\lambda}^{w'} f(\xi) \text{ for } \lambda \in \mathbb{C}^* \right\} \\ &= \left\{ f \in \mathcal{E}(w+w') : \left(e^{i\phi}\right)^* f(\xi) = e^{i\phi(w-w')} f(\xi) \right\}\end{aligned}$$

We will primarily be concerned with functions in

$$\mathcal{E}(w,w) = \left\{ f \in \mathcal{E}(w+w') : \left(e^{i\phi}\right)^* f(\xi) = f(\xi) \right\}$$

which descend to smooth functions on M.

For $k \in \mathbb{Z}$, we define

$$\begin{aligned} P_{w,w'} &: \mathcal{E}(w,w') \to \mathcal{E}(w-k, w'-k) \\ f &\mapsto 2^{-k} \mathcal{P}_k f, \end{aligned}$$

where $\mathcal{P}_k$ is defined in (4.1). Then choosing $w = w' = (k-(n+1))/2$, we get operators P_k defined on $\mathcal{E}(-N/2+k)$.which are invariant under the circle action $\left(e^{i\phi}\right)^*$ and hence may be viewed as smooth sections of a density bundle over M. These operators P_k are the CR-covariant differential operators which we will connect to poles of the scattering operator.

If X admits a globally defined approximate solution φ of the Monge-Ampère equation, then for each $p \in M = \partial X$ there is a neighborhood U of p and holomorphic coordinates $(z_1, \cdots, z_m)$ near p so that φ is an approximate solution of the Monge-Ampère equation in U. Let

$$\theta = \frac{i}{2}\left(\overline{\partial} - \partial\right)\varphi\Big|_M$$

be the induced pseudo-Hermitian structure on M, and let $\zeta = dz^1 \wedge \cdots \wedge dz^m\big|_M$. Then θ is volume-normalized with respect to ζ.

Let us denote by z_0 the induced fibre coordinate of $(\mathcal{K}^*)^{1/(n+2)}$ and let

$$Q = |z_0|^2 \varphi$$

Then Q is a globally defined smooth function on $\widetilde{\mathcal{G}}$ (which is diffeomorphic to $\mathbb{C} \times N$ for a collar neighborhood N of M in X) and the ambient metric on $\widetilde{\mathcal{G}}$ is the Kähler metric associated to the Kähler form

$$\omega = i\partial\overline{\partial}Q$$

where the corresponding metric g_θ on $\mathcal{C}$ is given by (4.3). The key computation linking the GJMS operators to the Laplacian is given in Proposition 5.4 of [9] and clearly generalizes to our situation. Thus we have:

Proposition 4.2. *If u is a smooth function on X then*

$$\widetilde{\Delta}\left(|z_0|^{2w}\varphi^w u\right) = \left(|z_0|^{2w}\varphi^w\right)\left(\Delta_\varphi + w(n+1+w)\right)u$$

where g is the metric associated to the Kähler form

$$\omega_\varphi = \frac{i}{2}\partial\overline{\partial}\log(-1/\varphi)$$

5. Proofs of the Main Theorems

Finally, we prove Theorems 1.1, 1.4, and 1.5.

Proof of Theorem 1.1. The statement about the poles of $S_X(s)$ and $s = m/2 + k/2$ is proved in Proposition 3.12. If g is a metric on X associated to the Kähler form $\omega = i\overline{\partial}\partial \log(-1/\varphi)$ for a globally defined approximate solution of the Monge-Ampère equation, then the identification of the residues of $S_X(s)$ with the CR-covariant differential operators of Fefferman and Hirachi is a consequence of Proposition 4.2 and the second characterization of the GJMS operators given in section 4.1. □

Proof of Theorem 1.4. Owing to Proposition 3.13, it suffices to identify $\lim_{s\to m} p_{m,s}1$ with the CR Q-curvature. This is a consequence of Remark 3.14. □

Proof of Theorem 1.5. To prove Theorem 1.5, let

$$u_s = \mathcal{P}(s)1$$

for s real. Observe that u_s is real and that $u_s \to 1$ as $s \to m$ uniformly on compact subsets of X. On the other hand, for $s \neq m$ but s close to m, u_s takes the form

$$u_s \sim x^{m-s}F(s) + x^s G(s) \tag{5.1}$$

where $F(s)$ and $G(s)$ belong to $\mathcal{C}^\infty(X)$, $G(s) = S_X(s)1 + \mathcal{O}(x)$ uniformly in s near m, and

$$F(s) - 1 \sim \sum_{k\geq 1} x^k F_k(s) \tag{5.2}$$

where $F_k \in \mathcal{C}^\infty(M)$ and $F_k(s) \to 0$ as $s \to m$, save for the $F_m(s)$ term which obeys

$$F_m(s) + S_X(s)1 \to 0 \text{ as } s \to m \tag{5.3}$$

(see the proof of Proposition 3.13). Note that

$$\int_M F_m(s)\psi = \int p_{m,s}1\ \psi.$$

As in [14] we will prove Theorem 1.5 by computing

$$\operatorname*{FP}_{\varepsilon\downarrow 0}\left(\int_{x>\varepsilon}\left[|du_s|^2 - s(m-s)u_s u_s\right]\omega^m\right)$$

(where FP denotes the Hadamard finite part) in two different ways. First, we use Proposition 3.3 with $u_1 = u_2 = u_s$ to conclude that

$$\operatorname*{FP}_{\varepsilon\downarrow 0}\left(\int_{x>\varepsilon}\left[|du_s|^2 - s(m-s)u_s u_s\right]\omega^m\right) = -m\int_M S_X(s)1\ \psi \tag{5.4}$$

Secondly (and somewhat more painfully), we use the asymptotic expansion of u_s and the asymptotic form of the volume form ω^m directly to conclude that

$$\operatorname*{FP}_{\varepsilon\downarrow 0}\left(\int_{x>\varepsilon}\left[|du_s|^2 - s(m-s)u_s u_s\right]\omega^m\right) = \frac{m}{2}L \tag{5.5}$$

from which the desired equality will follow. The computations follow along the lines of [14] with some trivial differences in the computation for (A.4) owing to the different form of the Laplacian on a complex manifold. We give a summary in Appendix A. □

Appendix A. CR Q-curvature and Asymptotic Volume

The purpose of this appendix is to summarize the calculations leading to the identity (5.5) used in the proof of Theorem 1.5. We will show that

$$\lim_{s\to m} s(m-s)\,\underset{\varepsilon\downarrow 0}{\mathrm{FP}}\left(\int_{x>\varepsilon} u_s^2\ \omega^m\right) = -mc_m \int_M Q_\theta^{CR}\ \psi + mL/2 \tag{A.1}$$

and

$$\lim_{s\to m}\underset{\varepsilon\downarrow 0}{\mathrm{FP}}\left(\int_\varepsilon^{x_0} |du_s|^2\ \ \omega^m\right) = -mc_m \int_M Q_\theta^{CR}\ \psi \tag{A.2}$$

As in [14], since $u_s \to 1$ uniformly on compacts of X, it suffices to compute the respective limits

$$\lim_{s\to m}\left[s(m-s)\,\underset{\varepsilon\downarrow 0}{\mathrm{FP}}\left(\int_{\varepsilon<x<x_0} u_s^2\ \omega^m\right)\right] \tag{A.3}$$

and

$$\lim_{s\to m}\left[\underset{\varepsilon\downarrow 0}{\mathrm{FP}}\left(\int_{\varepsilon<x<x_0} |du_s|^2\ \ \omega^m\right)\right] \tag{A.4}$$

for any $x_0 > 0$. This reduction allows us to use boundary coordinates and introduce asymptotic expansions for u_s and ω^m. In the computations we make use of the simple formulas

$$\underset{\varepsilon\downarrow 0}{\mathrm{FP}}\int_\varepsilon^{x_o} x^{m-2s+j}\,\frac{dx}{x} = \frac{x_0^{m-2s+j}}{m-2s+j} \tag{A.5}$$

(note that the finite part is independent of x_0 if $j = m$) and

$$\underset{\varepsilon\downarrow 0}{\mathrm{FP}}\int_\varepsilon^{x_0} \frac{dx}{x} = \log x_0 \tag{A.6}$$

Setting $x = -\varphi$, we also have from (2.14) that

$$\omega_\varphi^m = \frac{\eta}{x^m}\frac{dx}{x} \wedge (d\theta)^n \wedge \theta$$

for $\eta \in \mathcal{C}^\infty(X)$ with $\psi = \left(\eta|_M\right)(d\theta)^n \wedge \theta$ (here $\eta|_M = m/2^{n-1}$ in accordance with (2.16)). We will write

$$\eta \sim \sum_{k\geq 0} x^k \eta_k$$

for $\eta_k \in \mathcal{C}^\infty(M)$, so that

$$L = \int_M \eta_m\ \ (d\theta)^n \wedge \theta.$$

First, we consider (A.3). In expanding the density

$$u_s^2\ \omega^m = f_1 \frac{dx}{x} \wedge (d\theta)^n \wedge \theta$$

asymptotically in x, we may neglect terms which are integrable, or terms which give rise to finite parts which are holomorphic at $s = m$. It suffices then to compute the coefficient of x^{2m-2s} in the expansion for f_1, since only the x^{2m-2s} term will

give rise to a finite part with pole at $s = m$. Note that the resulting residue is independent of x_0 (see (A.5)). Since

$$\begin{aligned} u_s^2 &= x^{2m-2s}\left[1 + 2(F(s)-1) + (F(s)-1)^2\right] \\ &\quad + 2x^{2m}F(s)G(s) + x^{2s}G(s)^2 \end{aligned}$$

it suffices to examine the terms $x^{2m-2s}\eta_m$ and $2x^{2m-2s}F_m(s)$ in f_1. The first of these contributes $(m/2)L$ to (A.3) and the second contributes

$$-\int_M S_X(m)1\ \psi = -mc_m \int_M Q_\theta^{CR}\ \psi.$$

This leads to (A.1) as claimed

Next, we consider (A.4). From (2.18), (2.22), and the fact that

$$|u_m|^2 = \frac{1}{4}\left|(N - iT)\, u\right|^2$$

it follows that the density

$$|du_s|^2\,\omega^m = \frac{1}{1+rx}\left|x\partial_x u_s\right|^2\omega^m + xH^{\alpha\overline{\beta}}(u_s)_\alpha(u_s)_{\overline{\beta}}\omega^m \tag{A.7}$$

for a tensor $H^{\alpha\overline{\beta}}$ which is smooth in x down to $x = 0$. We will show that the first right-hand term in (A.7) leads to the right-hand side of (A.2) and the second right-hand term in (A.7) makes no contribution.

From the asymptotic form of u_s (see (5.1), (5.2), 5.3)) we have

$$x\partial_x u_s = x^{m-s}K_1(s) + x^s K_2(s) \tag{A.8}$$

where

$$K_1(s) = (m-s)F(s) + xF_x(s)$$

and $K_1(s)$ and $K_2(s)$ both approach zero as $s \to m$. For this reason, writing

$$|x\partial_x u_s|^2\,\omega^m = f_2\frac{dx}{x}\wedge(d\theta)^n\wedge\theta$$

we need only consider the coefficient of x^{2m-2s} in the expansion of f_2 since all other terms give rise to terms whose finite parts vanish as $s \to m$. On squaring (A.8) we have

$$|x\partial_x u_s|^2 = x^{2m-2s}K_1(s)^2 + 2x^m K_1(s)K_2(s) + x^{2s}K_2(s)^2.$$

We can drop terms containing $(m-s)^2$ times a holomorphic function since these will vanish as $s \to m$, even if the power x^{2s-2m} occurs in f_2. Thus we need to compute the coefficient of x^m in $K_1(s)^2$ to order $(m-s)$. This is $2(2m-s)(m-s)F_m(s)$ which contributes $-(2m-s)\int_M F_m(s)\ \psi$ to the finite part of $\int_{x>\varepsilon}|x\partial_x u_s|^2\,\omega^m$ and approaches $-mc_m\int_M Q_\theta^{CR}\ \psi$ as $s \to m$.

It remains to show that

$$\lim_{s\to m}\ \operatorname*{FP}_{\varepsilon\downarrow 0}\left(\int_\varepsilon^{x_0}\frac{rx}{1+rx}\left|x\partial_x u_s\right|^2\ \omega^m\right) = 0 \tag{A.9}$$

and

$$\lim_{s\to m}\ \operatorname*{FP}_{\varepsilon\downarrow 0}\left(\int_\varepsilon^{x_0} xH^{\alpha\overline{\beta}}(u_s)_\alpha(u_s)_{\overline{\beta}}\ \omega^m\right) = 0 \tag{A.10}$$

In the first case, we can use the analysis above to show that the coefficient of x^{2m-2s} in the expansion for $\left[(rx)/(1+rx)\right]\left|x\partial_x u_s\right|^2\ \omega^m$ vanishes as $(s-m)^2$ when $s \to m$,

implying (A.9). Introducing boundary local coordinates y_j, to prove that (A.10) holds it suffices to show that

$$\lim_{s\to m} \operatorname*{FP}_{\varepsilon\downarrow 0} \left(\int_{\varepsilon}^{x_0} x\phi \frac{\partial u}{\partial y_j} \ \frac{\partial u}{\partial y_k} \omega^m \right) = 0$$

where ϕ is a smooth function supported in a local coordinate patch near the boundary M. This follows from the fact that, in local coordinates (x, y) on X in a neighborhood of M,

$$\frac{\partial u}{\partial y_j} = x^{m-s} L_1(s) + x^s L_2(s)$$

where both $L_1(s)$ and $L_2(s)$ vanish to order $(m - s)$ as $s \to m$.

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